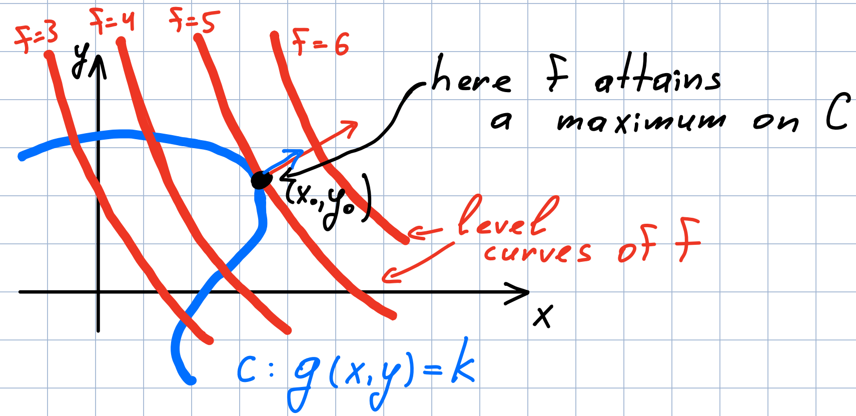
Last time:

Problem:

Find extreme values of $f(x,y)$ subject to constraint $g(x,y)=k$



level curves $g(x,y)=k$ and $f(x,y)=5$ touch at (x_0,y_0)

$$\Rightarrow \nabla f(x_0,y_0) = \underbrace{\lambda}_{} \nabla g(x_0,y_0)$$

↑ a number, "Lagrange multiplier"

Method of Lagrange multipliers

To find maximum & minimum values of $f(x,y)$

subject to constrain $g(x,y)=k$:

ⓐ find all values of x, y, λ such that

- $\nabla f(x,y) = \lambda \nabla g(x,y)$ $\quad \boxed{\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \end{aligned}}$
- $g(x,y) = k$

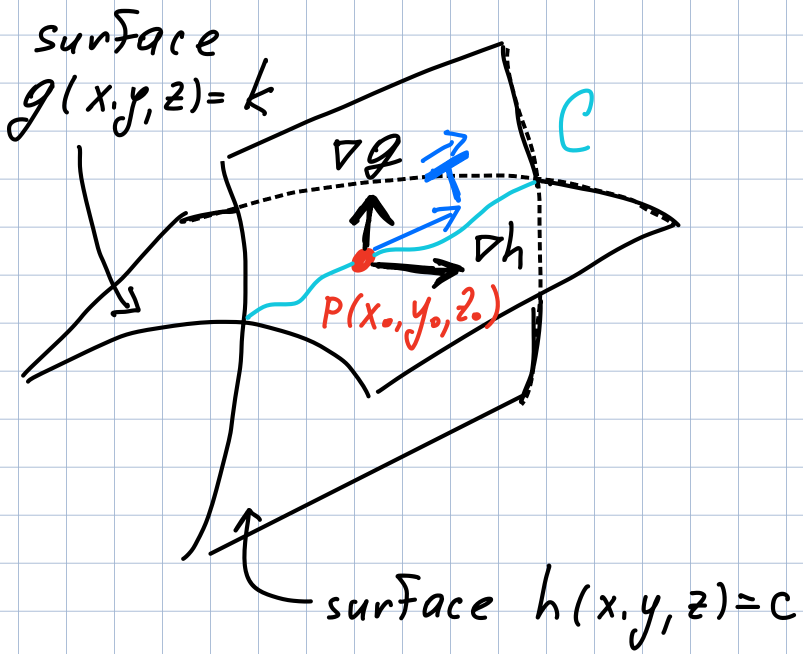
ⓑ evaluate f at all points found in (a)

maximum = largest value of f

minimum = smallest

Lagrange multipliers with two constraints

Problem: Find min/max values of $f(x,y,z)$ subject to two constraints: $\begin{cases} g(x,y,z)=k \\ h(x,y,z)=c \end{cases}$ define a curve C



If at $P(x_0,y_0,z_0)$ on C, f has a min or max, then directional derivative

$$D_{\vec{T}} f = 0, \qquad (D_{\vec{T}} f = \nabla f \cdot \vec{T})$$

where $\vec{T}$ is the unit tangent vector along C at P.

So, $\nabla f(x_0,y_0,z_0)$ is orthogonal to $\vec{T}$

$\Rightarrow \nabla f|_P$ lies in the plane orthogonal to C at P.

This plane is defined by $\nabla g(x_0,y_0,z_0)$ and $\nabla h(x_0,y_0,z_0)$.

Therefore

$$\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0) + \mu \nabla h(x_0,y_0,z_0)$$

Two Lagrange multipliers

Method of Lagrange multipliers

To find maximum & minimum values of $f(x,y,z)$

subject to constraints $g(x,y,z)=k$ and $h(x,y,z)=c$

(a) find all values of x, y, z, λ, μ such that

- $\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$
- $g(x,y,z) = k$
- $h(x,y,z) = c$

$$\boxed{\begin{aligned} f_x &= \lambda g_x + \mu h_x \\ f_y &= \lambda g_y + \mu h_y \\ f_z &= \lambda g_z + \mu h_z \end{aligned}}$$

(b) evaluate f at all points found in (a)

maximum = largest value of f

minimum = smallest

Ex: find the max value of $f(x,y,z) = x + 2y + 3z$ on the curve C of intersection of $\underbrace{x - y + z = 1}_{g(x,y,z)}$ (a plane) and $\underbrace{x^2 + y^2 = 1}_{h(x,y,z)}$ (a cylinder)

Sol:

$$\begin{cases} f_x = 1 = \lambda \cdot 1 + \mu \cdot 2x \\ f_y = 2 = \lambda \cdot (-1) + \mu \cdot 2y \\ f_z = 3 = \lambda \cdot 1 + \mu \cdot 0 \\ x - y + z = 1 \\ x^2 + y^2 = 1 \end{cases} \Rightarrow \lambda = 3 \quad \text{and} \quad \begin{cases} 3 + 2\mu x = 1 \\ -3 + 2\mu y = 2 \end{cases} \Rightarrow \begin{cases} x = -\frac{1}{\mu} \\ y = \frac{5}{2\mu} \end{cases}$$

Now, $x^2 + y^2 = 1 \Rightarrow \left(-\frac{1}{\mu}\right)^2 + \left(\frac{5}{2\mu}\right)^2 = 1$

$$\Rightarrow \frac{29}{4\mu^2} = 1 \Rightarrow \mu = \pm \frac{\sqrt{29}}{2}$$

Solution 1: $\lambda = 3$, $\mu = \frac{\sqrt{29}}{2}$, $x = -\frac{2}{\sqrt{29}}$, $y = \frac{5}{\sqrt{29}}$, $z = 1 - x + y = 1 + \frac{7}{\sqrt{29}}$

Solution 2: $\lambda = 3$, $\mu = -\frac{\sqrt{29}}{2}$, $x = \frac{2}{\sqrt{29}}$, $y = -\frac{5}{\sqrt{29}}$, $z = 1 - x + y = 1 - \frac{7}{\sqrt{29}}$

$f(x, y, z) = x + 2y + 3z$

$$f\left(-\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, 1 + \frac{7}{\sqrt{29}}\right) = \boxed{3 + \sqrt{29}}$$ max value on C

$$f\left(\frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}, 1 - \frac{7}{\sqrt{29}}\right) = 3 - \sqrt{29}$$

Ex: Plane $4x-3y+8z=5$ intersects the cone $z^2=x^2+y^2$ in an ellipse C. Find highest and lowest points on C.

Sol: $g(x,y,z)=4x-3y+8z$ $\qquad f(x,y,z)=z$

$h(x,y,z)=x^2+y^2-z^2$

WANTED: max/min of f subject to $g(x,y,z)=5$ and $h(x,y,z)=0$

$$\begin{cases} 0=\lambda\cdot 4+\mu\cdot 2x & \Rightarrow\ x=-\frac{4\lambda}{2\mu} \\ 0=\lambda\cdot(-3)+\mu\cdot 2y & \Rightarrow\ y=\frac{3\lambda}{2\mu} \\ 1=\lambda\cdot 8+\mu(-2z) & \Rightarrow\ z=\frac{8\lambda-1}{2\mu} \\ 4x-3y+8z=5 \\ x^2+y^2-z^2=0 \end{cases}$$

or $\mu=0$

if $\mu=0 \Rightarrow \lambda=0$ but then $1\neq 8\lambda-2\mu z$ so, $\mu\neq 0$.

$$4x-3y+8z=\frac{-16\lambda-9\lambda+64\lambda-8}{2\mu}=5 \Rightarrow 39\lambda-8=10\mu \Rightarrow \mu=\frac{39\lambda-8}{10}$$

$$x^2+y^2-z^2=\frac{1}{4\mu^2}\left(\underbrace{16\lambda^2+9\lambda^2-64\lambda^2+16\lambda-1}_{-39\lambda^2+16\lambda-1=0}\right)=0$$

$$-39\lambda^2+16\lambda-1=0 \Rightarrow \lambda=\frac{16\pm\sqrt{16^2-4\cdot 39}}{2\cdot 39}=\frac{16\pm 10}{78}$$

$\lambda_1 = \frac{6}{78} = \frac{1}{13}$; $\lambda_2 = \frac{26}{78} = \frac{1}{3}$

Solution 1: $\lambda = \frac{1}{13}$, $\mu = \frac{1}{10}\left(\frac{39}{13} - 8\right) = -\frac{1}{2}$, $x = -\frac{4\lambda}{2\mu} = \underline{\frac{4}{13}}$, $y = \frac{3\lambda}{2\mu} = \underline{-\frac{3}{13}}$, $z = \frac{2\lambda - 1}{2\mu} = \underline{\frac{5}{13}}$

lowest point on C

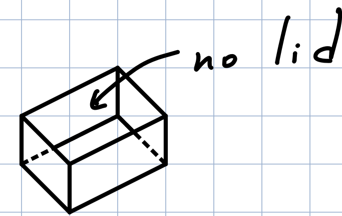
Solution 2: $\lambda = \frac{1}{3}$, $\mu = \frac{1}{2}$, $\boxed{x = -\frac{4}{3},\ y = 1,\ z = \frac{5}{3}}$

highest point on C

Ex: Rectangular box <u>without a lid</u> is made of $12\,m^2$ of cardboard. What is the max possible volume?

Sol: $x, y, z > 0$ $\quad f(x,y,z) = xyz$ - volume

$$g(x,y,z) = \underbrace{xy + 2xz + 2yz}_{\text{surface area}} = 12$$



Single constraint!

$$\begin{cases} yz = \lambda(y + 2z) \xrightarrow{\cdot \frac{1}{yz}} 1 = \lambda\left(\frac{1}{z} + \frac{2}{y}\right) & (1) \\ xz = \lambda(x + 2z) \xrightarrow{\cdot \frac{1}{xz}} 1 = \lambda\left(\frac{1}{z} + \frac{2}{x}\right) & (2) \\ xy = \lambda(2x + 2y) \xrightarrow{\cdot \frac{1}{xy}} 1 = \lambda\left(\frac{2}{y} + \frac{2}{x}\right) & (3) \\ xy + 2xz + 2yz = 12 \end{cases}$$

Note that $\lambda \neq 0$

① − ②: $2\lambda\left(\frac{1}{y}-\frac{1}{x}\right)=0 \Rightarrow y=x$ $\quad 1=\lambda\left(\frac{1}{z}+\frac{2}{y}\right) \Rightarrow 1=\lambda\frac{4}{x} \Rightarrow \lambda=\frac{x}{4}$

① − ③: $\lambda\left(\frac{1}{z}-\frac{2}{x}\right)=0 \Rightarrow z=\frac{x}{2}$ $\quad xy+2xz+2yz=12 \Rightarrow 3x^2=12$

$\Rightarrow x=\pm 2$

should be $>0 \Rightarrow 2$

$$\boxed{\begin{array}{l} x=2,\quad y=2,\quad z=1 \\ \text{max volume:}\quad 2\cdot 2\cdot 1=\underline{\underline{4}} \end{array}}$$